THE YONEDA LEMMA

or why we know what a black hole is without being able to see it

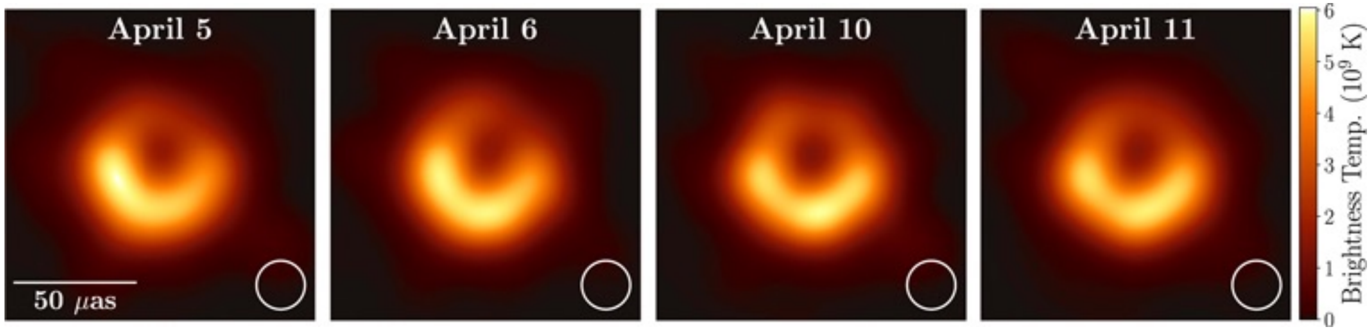
BSSM 2022

29/08/2022

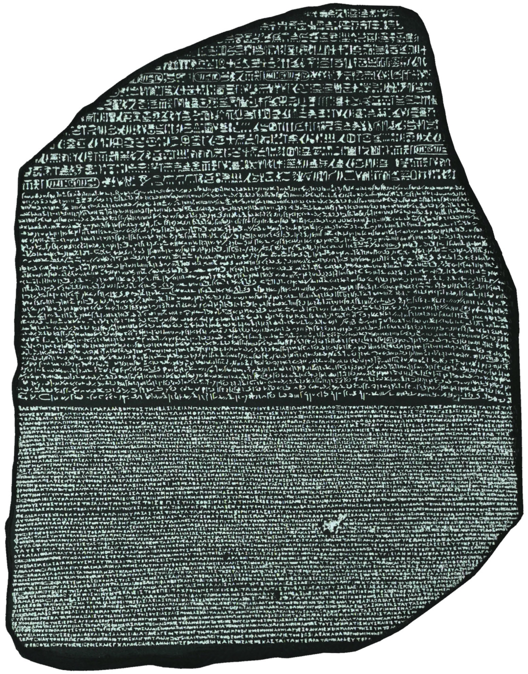
Julia Ramos González

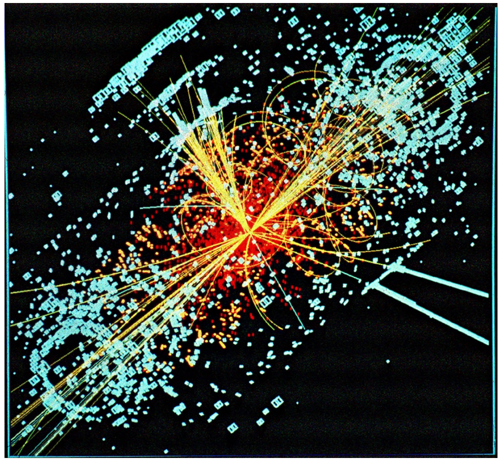
UCLouvain

Understanding the inaccessible



The Event Horizon Telescope Collaboration *et al* 2019 *ApJL* **875** L4 (License: CC-BY-3.0, unmodified image)





© 1997-2022 CERN (License: CC-BY-SA-4.0, unmodified image)
http://cdsweb.cern.ch/record/628469

Categories, our mathematical contexts

What is a CATEGORY?

Categories, our mathematical contexts

What is a CATEGORY?

$\Bigg\{$ a collection of objects $A, B, C,$

Categories, our mathematical contexts

What is a CATEGORY?

$\begin{cases} \text{a collection of objects } A, B, C, \\ \text{a collection of morphisms } f, g, h, \end{cases}$ such that

-
-
-

Categories, our mathematical contexts

What is a CATEGORY?

$\begin{cases} \text{a collection of objects } A, B, C, \\ \text{a collection of morphisms } f, g, h, \end{cases}$ such that

- each morphism has a specified domain and codomain

$$f \; A \longrightarrow B$$

-

-

Categories, our mathematical contexts

What is a CATEGORY?

$\begin{cases} \text{a collection of objects } A, B, C, \\ \text{a collection of morphisms } f, g, h, \end{cases}$ such that

- each morphism has a specified domain and codomain

$$f \; A \longrightarrow B$$

- for each object we have an identity morphism

$$1_A \; A \longrightarrow A$$

-

Categories, our mathematical contexts

What is a CATEGORY?

a collection of objects A, B, C,

a collection of morphisms f, g, h,

Such that

- each morphism has a specified domain and codomain

$$f \quad A \longrightarrow B$$

- for each object we have an identity morphism

$$1_A \quad A \longrightarrow A$$

- for each pair $f \ A \rightarrow B$, $g \ B \longrightarrow C$, we have a composite

$$gf \quad A \longrightarrow C$$

subject to the following axioms {

Categories, our mathematical contexts

What is a CATEGORY?

a collection of objects $A, B, C,$
a collection of morphisms $f, g, h,$
such that

- each morphism has a specified domain and codomain

$$f\ A \longrightarrow B$$

- for each object we have an identity morphism

$$1_A\ A \longrightarrow A$$

- for each pair $f\ A \rightarrow B$, $g\ B \longrightarrow C$, we have a composite

$$g f\ A \longrightarrow C$$

subject to the following axioms

given $f\ A \rightarrow B$, we have

$$f 1_A = f = 1_B f$$

Categories, our mathematical contexts

What is a CATEGORY?

a collection of objects $A, B, C,$
a collection of morphisms $f, g, h,$
such that

- each morphism has a specified domain and codomain

$$f\ A \longrightarrow B$$

- for each object we have an identity morphism

$$1_A\ A \longrightarrow A$$

- for each pair $f\ A \to B$, $g\ B \longrightarrow C$, we have a composite

$$gf\ A \longrightarrow C$$

subject to the following axioms

given $f\ A \to B$, we have

$$f 1_A = f = 1_B f$$

given $f\ A \to B$, $g\ B \to C$, $h\ C \to D$

$$h(gf) = (hg)f$$

Examples of categories

- Set
  - → Objects sets
  - → Morphisms Set(A, B) = functions from A to B
  - → composition of functions & identity functions
-
-

Examples of categories

- Set
  - → Objects sets
  - → Morphisms Set(A, B) = functions from A to B
  - → composition of functions & identity functions
- Grp
  - → Objects groups
  - → Morphisms Grp(G, H) = group homomorphisms G → H
  - → composition of homomorphisms & identity
-

Examples of categories

- Set → Objects sets
  → Morphisms Set(A, B) = functions from A to B
  → composition of functions & identity functions
- Grp → Objects groups
  → Morphisms Grp(G, H) = group homomorphisms $G \to H$
  → composition of homomorphisms & identity
- Top → Objects topological spaces
  → Morphisms Top(X, Y) = continuous maps $X \longrightarrow Y$
  → Composition of continuous maps & identities

Examples of categories

- $\mathbb{1}$ → Objects a single object $*$

  → Morphisms $\mathbb{1}(*,*) = \{1_* \; * \longrightarrow *\}$

-

-

Examples of categories

- $\mathbb{1}$ → Objects a single object $*$

  → Morphisms $\mathbb{1}(*,*) = \{ 1_* \; * \longrightarrow * \}$

Let G be a group

- $\underline{G}$ → Objects a single object $*$

  → Morphisms $\underline{G}(*,*) = G$

  → composition group operation $* \xrightarrow{a} * \xrightarrow{b} * = * \xrightarrow{ba} *$

  Identity the neutral element $* \xrightarrow{1_G} *$

-

Examples of categories

- $\mathbb{1}$ → Objects a single object $*$

  → Morphisms $\mathbb{1}(*,*) = \{ 1_* \; * \longrightarrow * \}$

Let G be a group

- $\underline{G}$ → Objects a single object $*$

  → Morphisms $\underline{G}(*,*) = G$

  → composition group operation $* \xrightarrow{a} * \xrightarrow{b} * = * \xrightarrow{ba} *$

  Identity the neutral element $* \xrightarrow{1_G} *$

Let $\mathcal{A}$ be a category

- $\mathcal{A}^{op}$ → objects the same as $\mathcal{A}$

  → Morphisms $\mathcal{A}^{op}(A, A') = \mathcal{A}(A', A)$ + identity and composition of $\mathcal{A}$

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left] & A' \arrow[r] & A'' \end{tikzcd} = \begin{tikzcd} A & A' \arrow[l] & A'' \arrow[l] \arrow[ll, bend right] \end{tikzcd}$$

« Sameness » of objects

Let $\mathcal{A}$ be a category

-

-

-

« Sameness » of objects

Let $\mathcal{A}$ be a category

- A morphism $f\ A \longrightarrow B$ in $\mathcal{A}$ is called an isomorphism if there exists another morphism $f^{-1}\ B \longrightarrow A$ such that
  -
  -
-
-

« Sameness » of objects

Let $\mathcal{A}$ be a category

- A morphism $f\ A \longrightarrow B$ in $\mathcal{A}$ is called an isomorphism if there exists another morphism $f^{-1}\ B \longrightarrow A$ such that
  - $ff^{-1} = 1_B\ \ B \longrightarrow B$
  -
-
-

« Sameness » of objects

Let $\mathcal{A}$ be a category

- A morphism $f\ A \longrightarrow B$ in $\mathcal{A}$ is called an **isomorphism** if there exists another morphism $f^{-1}\ B \longrightarrow A$ such that
  - $ff^{-1} = 1_B \quad B \longrightarrow B$
  - $f^{-1}f = 1_A \quad A \longrightarrow A$

•

•

« Sameness » of objects

Let $\mathcal{A}$ be a category

- A morphism $f \ A \longrightarrow B$ in $\mathcal{A}$ is called an **isomorphism** if there exists another morphism $f^{-1} \ B \longrightarrow A$ such that
  - $ff^{-1} = 1_B \quad B \longrightarrow B$
  - $f^{-1}f = 1_A \quad A \longrightarrow A$

  such f^{-1} is unique and it is called the **inverse** of f
-
-

« Sameness » of objects

Let $\mathcal{A}$ be a category

- A morphism $f\ A \longrightarrow B$ in $\mathcal{A}$ is called an isomorphism if there exists another morphism $f^{-1}\ B \longrightarrow A$ such that
  - $ff^{-1} = 1_B\quad B \longrightarrow B$
  - $f^{-1}f = 1_A\quad A \longrightarrow A$

  such f^{-1} is unique and it is called the inverse of f
- Two objects A, B in $\mathcal{A}$ are isomorphic if there exists an isomorphism $f\ A \longrightarrow B$ connecting them
-

« Sameness » of objects

Let $\mathcal{A}$ be a category

- A morphism $f\ A \longrightarrow B$ in $\mathcal{A}$ is called an isomorphism if there exists another morphism $f^{-1}\ B \longrightarrow A$ such that
  - $ff^{-1} = 1_B \quad B \longrightarrow B$
  - $f^{-1}f = 1_A \quad A \longrightarrow A$

  Such f^{-1} is unique and it is called the inverse of f
- Two objects A, B in $\mathcal{A}$ are isomorphic if there exists an isomorphism $f\ A \longrightarrow B$ connecting them
- Isomorphic objects have the same viewpoint $\mathcal{A}(A,C) \overset{-\circ f^{-1}}{\cong} \mathcal{A}(B,C)$ for all C

« Sameness » of objects

Let $\mathcal{A}$ be a category

- A morphism $f\ A \longrightarrow B$ in $\mathcal{A}$ is called an isomorphism if there exists another morphism $f^{-1}\ B \longrightarrow A$ such that
  - $ff^{-1} = 1_B\quad B \longrightarrow B$
  - $f^{-1}f = 1_A\quad A \longrightarrow A$

  Such f^{-1} is unique and it is called the inverse of f
- Two objects A, B in $\mathcal{A}$ are isomorphic if there exists an isomorphism $f\ A \longrightarrow B$ connecting them
- Isomorphic objects have the same viewpoint $\mathcal{A}(A,C) \overset{-\circ f^{-1}}{\cong} \mathcal{A}(B,C)$ for all C

  and the rest of objects see them as the same $\mathcal{A}(C,A) \overset{f\circ -}{\cong} \mathcal{A}(C,B)$

Relating categories functors

Let $\mathcal{A}, \mathcal{B}$ be two categories A functor

$$F \ \mathcal{A} \longrightarrow \mathcal{B}$$

consists of

•

•

Relating categories functors

Let $\mathcal{A}, \mathcal{B}$ be two categories A functor

$$F\ \mathcal{A} \longrightarrow \mathcal{B}$$

consists of

- a function $\mathrm{obj}(\mathcal{A}) \longrightarrow \mathrm{obj}(\mathcal{B})$ $A \longmapsto F(A)$
-

Relating categories functors

Let $\mathcal{A}, \mathcal{B}$ be two categories A functor

$$F\ \mathcal{A} \longrightarrow \mathcal{B}$$

consists of

- a function $\mathrm{obj}(\mathcal{A}) \longrightarrow \mathrm{obj}(\mathcal{B})$ $\quad A \longmapsto F(A)$
- a function $\mathcal{A}(A, A') \longrightarrow \mathcal{B}(F(A), F(A'))$

$$f \longmapsto F(f)$$

Relating categories functors

Let $\mathcal{A}, \mathcal{B}$ be two categories A functor

$$F \ \mathcal{A} \longrightarrow \mathcal{B}$$

consists of

- a function $\mathrm{obj}(\mathcal{A}) \longrightarrow \mathrm{obj}(\mathcal{B})$ $\quad A \longmapsto F(A)$
- a function $\mathcal{A}(A, A') \longrightarrow \mathcal{B}(F(A), F(A'))$

$$f \longmapsto F(f)$$

subject to the following axioms

-
-

Relating categories functors

Let $\mathcal{A}, \mathcal{B}$ be two categories A functor

$$F \quad \mathcal{A} \longrightarrow \mathcal{B}$$

consists of

- a function $\mathrm{obj}(\mathcal{A}) \longrightarrow \mathrm{obj}(\mathcal{B}) \quad A \longmapsto F(A)$
- a function $\mathcal{A}(A, A') \longrightarrow \mathcal{B}(F(A), F(A'))$

$$f \longmapsto F(f)$$

subject to the following axioms

- for every $A \in \mathrm{obj}(\mathcal{A})$, $\quad F(1_A) = 1_{F(A)}$
-

Relating categories: functors

Let $\mathcal{A}, \mathcal{B}$ be two categories. A functor

$$F\ \mathcal{A} \longrightarrow \mathcal{B}$$

consists of

- a function $\operatorname{obj}(\mathcal{A}) \longrightarrow \operatorname{obj}(\mathcal{B})$ $\quad A \longmapsto F(A)$
- a function $\mathcal{A}(A, A') \longrightarrow \mathcal{B}(F(A), F(A'))$

$$f \longmapsto F(f)$$

subject to the following axioms

- for every $A \in \operatorname{obj}(\mathcal{A})$, $\quad F(1_A) = 1_{F(A)}$
- for every $A \xrightarrow{f} A' \xrightarrow{g} A''$ in $\mathcal{A}$, $\quad F(gf) = F(g)F(f)$

Examples of functors

- $U \quad \mathrm{Grp} \longrightarrow \mathrm{Set}$

  $\longmapsto$

  $\longmapsto$

-

Examples of functors

- $U \quad \mathbf{Grp} \longrightarrow \mathbf{Set}$

  $G \longmapsto U(G)$ = the underlying set of the group G

  $\longmapsto$

-

Examples of functors

- $U \quad \mathrm{Grp} \longrightarrow \mathrm{Set}$

  $G \longmapsto U(G)$ = the underlying set of the group G

  $[G \xrightarrow{f} H] \longmapsto [U(f) \ U(G) \longrightarrow U(H)]$ = the underlying function of the group homomorphism f

-

Examples of functors

- $U \quad \mathrm{Grp} \longrightarrow \mathrm{Set}$

  $G \longmapsto U(G)$ = the underlying set of the group G

  $\left[G \xrightarrow{f} H \right] \longmapsto \left[U(f) \quad U(G) \longrightarrow U(H) \right]$ = the underlying function of the group homomorphism f

- $F \quad \mathrm{Set} \longrightarrow \mathrm{Grp}$

  $\longmapsto$

  $\longmapsto$

Examples of functors

- $U \quad \mathrm{Grp} \longrightarrow \mathrm{Set}$

  $G \longmapsto U(G)$ = the underlying set of the group G

  $\left[G \xrightarrow{f} H\right] \longmapsto \left[U(f) \quad U(G) \longrightarrow U(H)\right]$ = the underlying function of the group homomorphism f

- $F \quad \mathrm{Set} \longrightarrow \mathrm{Grp}$

  $X \longmapsto F(X)$ = the free group generated by the set X

  $\longmapsto$

Examples of functors

- $U \quad \mathrm{Grp} \longrightarrow \mathrm{Set}$

  $G \longmapsto U(G)$ = the underlying set of the group G

  $[G \xrightarrow{f} H] \longmapsto [U(f) \quad U(G) \longrightarrow U(H)]$ = the underlying function of the group homomorphism f

- $F \quad \mathrm{Set} \longrightarrow \mathrm{Grp}$

  $X \longmapsto F(X)$ = the free group generated by the set X

  $[X \xrightarrow{f} Y] \longmapsto [F(f) \quad F(X) \longrightarrow F(Y)]$ = the group homomorphism induced by f

Examples of functors

- $U \quad \mathbf{Grp} \longrightarrow \mathbf{Set}$

  $G \longmapsto U(G)$ = the underlying set of the group G

  $\left[G \xrightarrow{f} H\right] \longmapsto \left[U(f) \; U(G) \longrightarrow U(H)\right]$ = the underlying function of the group homomorphism f

- $F \quad \mathbf{Set} \longrightarrow \mathbf{Grp}$

  $X \longmapsto F(X)$ = the free group generated by the set X

  $\left[X \xrightarrow{f} Y\right] \longmapsto \left[F(f) \; F(X) \longrightarrow F(Y)\right]$ = the group homomorphism induced by f

  $x_1 x_2 \longmapsto f(x_1) f(x_2)$

Examples of functors

- $H_n \quad \mathrm{Top} \longrightarrow \mathrm{Ab}$

  category of abelian groups with group homomorphisms

  $\longmapsto$

  $\longmapsto$

-

Examples of functors

- $H_n \quad \mathrm{Top} \longrightarrow \mathrm{Ab}$

  category of abelian groups with group homomorphisms

  $X \longmapsto H_n(X)$ = nth homology group

  $\longmapsto$

-

Examples of functors

category of abelian groups with group homomorphisms

- $H_n \quad \mathrm{Top} \longrightarrow \mathrm{Ab}$

$$X \longmapsto H_n(X)$$ = nth homology group

$$[X \xrightarrow{f} Y] \longmapsto [H_n(f)\ H_n(X) \longrightarrow H_n(Y)]$$ = induced morphism in homology

-

Examples of functors

- $H_n \quad \mathrm{Top} \longrightarrow \mathrm{Ab}$ ↙ category of abelian groups with group homomorphisms

  $X \longmapsto H_n(X)$ = nth homology group

  $[X \xrightarrow{f} Y] \longmapsto [H_n(f) \; H_n(X) \longrightarrow H_n(Y)]$ = induced morphism in homology

- $P \quad \mathrm{Set} \longrightarrow \mathrm{Set}$

  $\longmapsto$

  $\longmapsto$

Examples of functors

- $H_n \quad \mathbf{Top} \longrightarrow \mathbf{Ab}$

  category of abelian groups with group homomorphisms

  $X \longmapsto H_n(X)$ = nth homology group

  $[X \xrightarrow{f} Y] \longmapsto [H_n(f) \ H_n(X) \longrightarrow H_n(Y)]$ = induced morphism in homology

- $P \quad \mathbf{Set} \longrightarrow \mathbf{Set}$

  $X \longmapsto P(X)$ = set of subsets of X

  $\longmapsto$

Examples of functors

- H_n Top $\longrightarrow$ Ab ↙ category of abelian groups with group homomorphisms

$$X \longmapsto H_n(X) = n^{th} \text{ homology group}$$

$$[X \xrightarrow{f} Y] \longmapsto [H_n(f)\ H_n(X) \longrightarrow H_n(Y)] = \text{induced morphism in homology}$$

- P Set $\longrightarrow$ Set

$$X \longmapsto P(X) = \text{set of subsets of } X$$

$$[X \xrightarrow{f} Y] \longmapsto [P(f)\ P(X) \longrightarrow P(Y)] = \text{induced function between the power sets}$$

Examples of functors

- H_n Top $\longrightarrow$ Ab ↙ category of abelian groups with group homomorphisms

  $X \longmapsto H_n(X)$ = nth homology group

  $[X \xrightarrow{f} Y] \longmapsto [H_n(f)\ H_n(X) \longrightarrow H_n(Y)]$ = induced morphism in homology

- P Set $\longrightarrow$ Set

  $X \longmapsto P(X)$ = set of subsets of X

  $[X \xrightarrow{f} Y] \longmapsto [P(f)\ P(X) \longrightarrow P(Y)]$ = induced function between the power sets

  $X_1 \subseteq X \mapsto \{f(x)\}_{x \in X_1} \subseteq Y$

Examples of functors

Let G, H be groups

-

-

Examples of functors

Let G, H be groups

- $F \quad \underline{G} \longrightarrow \underline{H}$

$$\mathrm{obj}(\underline{G}) \longrightarrow \mathrm{obj}(\underline{H})$$

$$\underline{G}(*,*) \longrightarrow \underline{H}(*,*)$$

-

Examples of functors

Let G, H be groups

- $F \quad \underline{G} \longrightarrow \underline{H}$

  $\mathrm{obj}(\underline{G}) \longrightarrow \mathrm{obj}(\underline{H}) \qquad * \longmapsto *$

  $\underset{\substack{\parallel \\ G}}{\underline{G}(*,*)} \longrightarrow \underset{\substack{\parallel \\ H}}{\underline{H}(*,*)} \qquad g \longmapsto F(g) \quad \text{st} \quad \begin{cases} F(1_G) = 1_{F(*)} \\ F(gg') = F(g)\,F(g') \end{cases}$

•

Examples of functors

Let G, H be groups

- $F \quad \underline{G} \longrightarrow \underline{H}$

$$\mathrm{obj}(\underline{G}) \longrightarrow \mathrm{obj}(\underline{H}) \qquad * \longmapsto *$$

$$\underset{\overset{\|}{G}}{\underline{G}(*,*)} \longrightarrow \underset{\overset{\|}{H}}{\underline{H}(*,*)} \qquad g \longmapsto F(g) \quad \text{st} \quad \begin{cases} F(1_G) = 1_{F(*)} \\ F(gg') = F(g)\,F(g') \end{cases}$$

it is nothing but a group homomorphism!

-

Examples of functors

Let G, H be groups

- $F \quad \underline{G} \longrightarrow \underline{H}$

  $\mathrm{obj}(\underline{G}) \longrightarrow \mathrm{obj}(\underline{H}) \qquad * \longmapsto *$

  $$\begin{tikzcd} \underline{G}(*,*) \arrow[r] \arrow[d, equal] & \underline{H}(*,*) \arrow[d, equal] \\ G & H \end{tikzcd}$$

  $g \longmapsto F(g)$ st $\begin{cases} F(1_G) = 1_{F(*)} \\ F(gg') = F(g)F(g') \end{cases}$

  it is nothing but a group homomorphism!

- $F \quad \underline{G} \longrightarrow \underline{\mathrm{Set}}$

  $\mathrm{obj}(\underline{G}) \longrightarrow \mathrm{obj}(\underline{\mathrm{Set}})$

  $\underline{G}(*,*) \longrightarrow \underline{\mathrm{Set}}(F(*), F(*))$

Examples of functors

Let G, H be groups

- $F \quad \underline{G} \longrightarrow \underline{H}$

$\mathrm{obj}(\underline{G}) \longrightarrow \mathrm{obj}(\underline{H}) \qquad * \longmapsto *$

$\underset{\substack{\| \\ G}}{\underline{G}(*,*)} \longrightarrow \underset{\substack{\| \\ H}}{\underline{H}(*,*)} \qquad g \longmapsto F(g) \quad \text{s t} \quad \begin{cases} F(1_G) = 1_{F(*)} \\ F(gg') = F(g)\,F(g') \end{cases}$

it is nothing but a group homomorphism!

- $F \quad \underline{G} \longrightarrow \mathbf{Set}$

$\mathrm{obj}(\underline{G}) \longrightarrow \mathrm{obj}(\mathbf{Set}) \qquad * \longmapsto F(*)$

$G = \underline{G}(*,*) \longrightarrow \mathbf{Set}(F(*), F(*)) \quad g \longmapsto F(g) \quad \text{s t} \quad \begin{cases} F(1_G) = 1_{F(*)} \\ F(gg') = F(g)\,F(g') \end{cases}$

Examples of functors

Let G, H be groups

- $F \quad \underline{G} \longrightarrow \underline{H}$

$$\mathrm{obj}(\underline{G}) \longrightarrow \mathrm{obj}(\underline{H}) \qquad * \longmapsto *$$

$$\underset{\substack{\| \\ G}}{\underline{G}(*,*)} \longrightarrow \underset{\substack{\| \\ H}}{\underline{H}(*,*)} \qquad g \longmapsto F(g) \quad \text{st} \quad \begin{cases} F(1_G) = 1_{F(*)} \\ F(gg') = F(g)\,F(g') \end{cases}$$

it is nothing but a group homomorphism!

- $F \quad \underline{G} \longrightarrow \mathrm{Set}$

$$\mathrm{obj}(\underline{G}) \longrightarrow \mathrm{obj}(\mathrm{Set}) \qquad * \longmapsto F(*)$$

$$G = \underline{G}(*,*) \longrightarrow \mathrm{Set}(F(*), F(*)) \quad g \longmapsto F(g) \quad \text{st} \quad \begin{cases} F(1_G) = 1_{F(*)} \\ F(gg') = F(g)\,F(g') \end{cases}$$

it is nothing but a left G-set!

Hom-functors

Homs are sets

Let $\mathcal{A}$ be a locally small category

We define the functor $h^A\ \mathcal{A} \to \mathbf{Set}$ as follows

$$\mathrm{obj}(\mathcal{A}) \longrightarrow \mathrm{obj}(\mathbf{Set})$$

$$\longmapsto$$

$$\mathcal{A}(B,C) \longrightarrow \mathbf{Set}(\mathcal{A}(A,B), \mathcal{A}(A,C))$$

$$\longmapsto$$

Hom-functors

Homs are sets

Let $\mathcal{A}$ be a locally small category

We define the functor $h^A \; \mathcal{A} \to \mathbf{Set}$ as follows

$$
\begin{array}{ccc}
\mathrm{obj}(\mathcal{A}) & \longrightarrow & \mathrm{obj}(\mathbf{Set}) \\
B & \longmapsto & h^A(B) = \mathcal{A}(A, B) \\
 & & \\
\mathcal{A}(B, C) & \longrightarrow & \mathbf{Set}(\mathcal{A}(A, B), \mathcal{A}(A, C)) \\
 & \longmapsto &
\end{array}
$$

Hom-functors

Homs are sets

Let $\mathcal{A}$ be a locally small category

We define the functor $h^A\ \mathcal{A} \to \mathbf{Set}$ as follows

$$\mathrm{obj}(\mathcal{A}) \longrightarrow \mathrm{obj}(\mathbf{Set})$$

$$B \longmapsto h^A(B) = \mathcal{A}(A, B)$$

$$\mathcal{A}(B, C) \longrightarrow \mathbf{Set}(\mathcal{A}(A, B), \mathcal{A}(A, C))$$

$$[f\ B \to C] \longmapsto h^A(f)\ \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A, C)$$

$$\longmapsto$$

Hom-functors

Homs are sets

Let $\mathcal{A}$ be a locally small category

We define the functor $h^A\ \mathcal{A} \to \mathbf{Set}$ as follows

$$\mathrm{obj}(\mathcal{A}) \longrightarrow \mathrm{obj}(\mathbf{Set})$$

$$B \longmapsto h^A(B) = \mathcal{A}(A, B)$$

$$\mathcal{A}(B, C) \longrightarrow \mathbf{Set}(\mathcal{A}(A, B), \mathcal{A}(A, C))$$

$$[f\ B \to C] \longmapsto h^A(f)\ \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A, C)$$

$$[g\ A \to B] \longmapsto [fg\ A \to C]$$

Hom-functors

Homs are sets

Let $\mathcal{A}$ be a locally small category

We define the functor $h^A \; \mathcal{A} \to \mathbf{Set}$ as follows

$$\begin{aligned} \operatorname{obj}(\mathcal{A}) &\longrightarrow \operatorname{obj}(\mathbf{Set}) \\ B &\longmapsto h^A(B) = \mathcal{A}(A, B) \end{aligned}$$

$$\begin{aligned} \mathcal{A}(B, C) &\longrightarrow \mathbf{Set}(\mathcal{A}(A, B), \mathcal{A}(A, C)) \\ [f \; B \to C] &\longmapsto h^A(f) \; \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A, C) \\ & \qquad\qquad [g \; A \to B] \longmapsto [fg \; A \to C] \end{aligned}$$

h^A tells us what the object A sees from the category $\mathcal{A}$

Hom-functors

Homs are sets

Let $\mathcal{A}$ be a locally small category

We define the functor $h_A \ \mathcal{A}^{op} \to \mathbf{Set}$ as follows

$$\mathrm{obj}(\mathcal{A}^{op}) \longrightarrow \mathrm{obj}(\mathbf{Set})$$

$$B \longmapsto h_A(B) = \mathcal{A}(B, A)$$

$$\mathcal{A}^{op}(B, C) = \mathcal{A}(C, B) \longrightarrow \mathbf{Set}(\mathcal{A}(B, A), \mathcal{A}(C, A))$$

$$[f \ C \to B] \longmapsto h_A(f) \ \mathcal{A}(B, A) \longrightarrow \mathcal{A}(C, A)$$

$$[g \ B \to A] \longmapsto [gf \ C \to A]$$

Hom-functors

Homs are sets

Let $\mathcal{A}$ be a locally small category

We define the functor $h_A \; \mathcal{A}^{op} \to \mathbf{Set}$ as follows

$$\mathrm{obj}(\mathcal{A}^{op}) \longrightarrow \mathrm{obj}(\mathbf{Set})$$

$$B \longmapsto h_A(B) = \mathcal{A}(B, A)$$

$$\mathcal{A}^{op}(B, C) = \mathcal{A}(C, B) \longrightarrow \mathbf{Set}(\mathcal{A}(B, A), \mathcal{A}(C, A))$$

$$[f \; C \to B] \longmapsto h_A(f) \; \mathcal{A}(B, A) \longrightarrow \mathcal{A}(C, A)$$

$$[g \; B \to A] \longmapsto [gf \; C \to A]$$

h_A tells us what the category $\mathcal{A}$ sees of the object A

Preserving viewpoints

- A functor $F\ \mathcal{A} \to \mathcal{B}$ is called fully faithful if for every pair $A, B \in \mathrm{obj}(\mathcal{A})$ the functions

$$\mathcal{A}(A,B) \xrightarrow{F} \mathcal{B}(F(A), F(B))$$

are bijective

- Example

Preserving viewpoints

- A functor $F\ \mathcal{A} \longrightarrow \mathcal{B}$ is called fully faithful if for every pair $A, B \in \mathrm{obj}(\mathcal{A})$ the functions

$$\mathcal{A}(A,B) \xrightarrow{F} \mathcal{B}(F(A), F(B))$$

are bijective

- Example

$$Ab \xrightarrow{\iota} Grp$$

$$G \longmapsto \iota(G) = G$$

$$[G \xrightarrow{f} H] \longmapsto [\iota(f) = f\quad G \longrightarrow H]$$

(morphisms in Ab are the homomorphisms of groups)

Relating functors natural transformations

Let $\mathcal{A}, \mathcal{B}$ be two locally small categories

Let F, G be two functors $\mathcal{A} \longrightarrow \mathcal{B}$

-

Relating functors natural transformations

Let $\mathcal{A}, \mathcal{B}$ be two locally small categories

Let F, G be two functors $\mathcal{A} \longrightarrow \mathcal{B}$

- A natural transformation α $F \Rightarrow G$ is given by a morphism α_A $F(A) \longrightarrow G(A)$ in $\mathcal{B}$ such that for all f $A \longrightarrow A'$ in $\mathcal{A}$ the following diagram in $\mathcal{B}$ commutes

$$\begin{tikzcd}
F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(A') \arrow[r, "\alpha_{A'}"] & G(A')
\end{tikzcd}$$

Categories of functors

Let $\mathcal{A}, \mathcal{B}$ be two locally small categories

Let F, G, H be functors $\mathcal{A} \longrightarrow \mathcal{B}$

-

-

-

Categories of functors

Let $\mathcal{A}, \mathcal{B}$ be two locally small categories

Let F, G, H be functors $\mathcal{A} \longrightarrow \mathcal{B}$

- We can compose natural transformations

-

-

Categories of functors

Let $\mathcal{A}, \mathcal{B}$ be two locally small categories

Let F, G, H be functors $\mathcal{A} \longrightarrow \mathcal{B}$

- We can compose natural transformations

  $\alpha\ F \Rightarrow G \ , \quad \beta\ G \Rightarrow H \ \rightsquigarrow \ \beta\alpha\ F \Rightarrow H$ given by

  $(\beta\alpha)_A = \beta_A \alpha_A \ F(A) \longrightarrow H(A)$

-

-

Categories of functors

Let $\mathcal{A}, \mathcal{B}$ be two locally small categories

Let F, G, H be functors $\mathcal{A} \longrightarrow \mathcal{B}$

- We can compose natural transformations

  $\alpha \; F \Rightarrow G \;, \quad \beta \; G \Rightarrow H \rightsquigarrow \beta\alpha \; F \Rightarrow H$ given by

  $(\beta\alpha)_A = \beta_A \alpha_A \; F(A) \longrightarrow H(A)$

- There is an identity natural transformation

-

Categories of functors

Let $\mathcal{A}, \mathcal{B}$ be two locally small categories

Let F, G, H be functors $\mathcal{A} \longrightarrow \mathcal{B}$

- We can compose natural transformations

$$\alpha\ F \Rightarrow G\ ,\quad \beta\ G \Rightarrow H \rightsquigarrow \beta\alpha\ F \Rightarrow H \text{ given by}$$

$$(\beta\alpha)_A = \beta_A \alpha_A\ F(A) \longrightarrow H(A)$$

- There is an identity natural transformation

$$1_F\ F \Rightarrow F \text{ given by } (1_F)_A = 1_{F(A)}$$

-

Categories of functors

Let $\mathcal{A}, \mathcal{B}$ be two locally small categories

Let F, G, H be functors $\mathcal{A} \longrightarrow \mathcal{B}$

- We can compose natural transformations

$$\alpha\ F \Rightarrow G \ , \quad \beta\ G \Rightarrow H \ \rightsquigarrow \ \beta\alpha\ F \Rightarrow H \ \text{given by}$$

$$(\beta\alpha)_A = \beta_A \alpha_A \ \ F(A) \longrightarrow H(A)$$

- There is an identity natural transformation

$$1_F\ F \Rightarrow F \ \text{given by} \ (1_F)_A = 1_{F(A)}$$

- Therefore, we can form a category of functors $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ with objects the functors and morphisms the natural transformations

Examples of natural transformations

- $$\begin{tikzcd} \underline{G} \arrow[r, bend left=50, "X"{name=U}] \arrow[r, bend right=50, "Y"'{name=D}] & \mathrm{Set} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$

Examples of natural transformations

- $$\begin{tikzcd} \underline{G} \arrow[r, bend left=50, "X"{name=U}] \arrow[r, bend right=50, "Y"'{name=D}] & \mathrm{Set} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$

$\alpha_* \quad X \longrightarrow Y$ in Set st for all $g \in \overset{G}{\overset{\parallel}{\underline{G}(*,*)}}$

$$\begin{tikzcd} X \arrow[r, "\alpha_*"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ X \arrow[r, "\alpha_*"'] & Y \end{tikzcd}$$

Examples of natural transformations

- $$\begin{tikzcd} \underline{G} \arrow[r, bend left=50, "X"] \arrow[r, bend right=50, "Y"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathrm{Set} \end{tikzcd}$$

$\alpha_* \quad X \longrightarrow Y$ in Set st for all $g \in \overset{\substack{G \\ \shortparallel}}{\underline{G}(*,*)}$

$$\begin{tikzcd} X \arrow[r, "\alpha_*"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ X \arrow[r, "\alpha_*"'] & Y \end{tikzcd}$$

in other words, $\alpha_*(gx) = g\alpha_*(x) \quad \forall x \in X \quad \forall g \in G$

Examples of natural transformations

- $$\begin{tikzcd} \underline{G} \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathrm{Set} \end{tikzcd}$$

  $\alpha_* \; X \longrightarrow Y$ in Set st for all $g \in \overset{G}{\overset{\shortparallel}{\underline{G}}}(*,*)$
  $$\begin{tikzcd} X \arrow[r, "\alpha_*"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ X \arrow[r, "\alpha_*"'] & Y \end{tikzcd}$$

  in other words, $\alpha_*(gx) = g\,\alpha_*(x) \quad \forall\, x \in X \;\; \forall\, g \in G$

  $\rightsquigarrow$ this is nothing but a G-equivariant map of left G-sets!

Examples of natural transformations

- $$\begin{tikzcd} \underline{G} \arrow[r, bend left, "X"{name=U}] \arrow[r, bend right, "Y"'{name=D}] & \mathrm{Set} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$

  $\alpha_* \; X \longrightarrow Y$ in Set st for all $g \in \underline{G}(*,*)$ $\overset{G}{=}$

  $$\begin{tikzcd} X \arrow[r, "\alpha_*"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ X \arrow[r, "\alpha_*"'] & Y \end{tikzcd}$$

  in other words, $\alpha_*(gx) = g\alpha_*(x) \quad \forall x \in X \; \forall g \in G$

  ⇝ this is nothing but a G-equivariant map of left G-sets!

we have that $\mathrm{Fun}(\underline{G}, \mathrm{Set})$ = category of left G-sets + equiv maps

Examples of natural transformations

- $$\begin{tikzcd} \mathbb{1} \arrow[r, bend left, "A"{name=U}] \arrow[r, bend right, "B"'{name=D}] & \mathcal{A} \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$ $\alpha_* \; A \to B$ in $\mathcal{A}$ st $$\begin{tikzcd} A \arrow[r, "\alpha_*"] \arrow[d, "1_A"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "1_B"] \\ A \arrow[r, "\alpha_*"] & B \end{tikzcd}$$ ↙ vacuous condition

-

Examples of natural transformations

- $$\begin{tikzcd} \mathbb{1} \arrow[r, bend left, "A", ""{name=U, below}] \arrow[r, bend right, "B"', ""{name=D}] & \mathcal{A} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$ $\alpha_* \; A \to B$ in $\mathcal{A}$ st $$\begin{tikzcd} A \arrow[r, "\alpha_*"] \arrow[d, "1_A"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "1_B"] \\ A \arrow[r, "\alpha_*"] & B \end{tikzcd}$$ vacuous condition

$\rightsquigarrow$ this is nothing but a morphism in $\mathcal{A}$ $\Rightarrow$ $\mathrm{Fun}(\mathbb{1}, \mathcal{A}) = \mathcal{A}$

-

Examples of natural transformations

- $$\begin{tikzcd} \mathbb{1} \arrow[r, bend left=50, "A"{name=U}] \arrow[r, bend right=50, "B"'{name=D}] & \mathcal{A} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$ $\alpha_* \quad A \to B$ in $\mathcal{A}$ st $$\begin{tikzcd} A \arrow[r, "\alpha_*"] \arrow[d, "1_A"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "1_B"] \\ A \arrow[r, "\alpha_*"] & B \end{tikzcd}$$ ↙ vacuous condition

$\rightsquigarrow$ this is nothing but a morphism in $\mathcal{A}$ $\Rightarrow$ $\mathrm{Fun}(\mathbb{1}, \mathcal{A}) = \mathcal{A}$

- $$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left=50, "1_{\mathrm{Set}}"{name=U}] \arrow[r, bend right=50, "P"'{name=D}] & \mathrm{Set} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$

Examples of natural transformations

- $$\begin{tikzcd} \mathbb{1} \arrow[r, bend left=50, "A"{name=U}] \arrow[r, bend right=50, "B"'{name=D}] & \mathcal{A} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$ α_* $A \to B$ in $\mathcal{A}$ st $$\begin{tikzcd} A \arrow[r, "\alpha_*"] \arrow[d, "1_A"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "1_B"] \\ A \arrow[r, "\alpha_*"] & B \end{tikzcd}$$ ← vacuous condition

⇝ this is nothing but a morphism in $\mathcal{A}$ $\Rightarrow$ $Fun(\mathbb{1}, \mathcal{A}) = \mathcal{A}$

- $$\begin{tikzcd} Set \arrow[r, bend left=50, "1_{Set}"{name=U}] \arrow[r, bend right=50, "P"'{name=D}] & Set \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$ $$\begin{tikzcd} & X \arrow[d, equal] & & \\ \alpha_X & 1_{Set}(X) \arrow[r] & P(X) & \text{in } Set \\ & x \arrow[r, mapsto] & \{x\} & \end{tikzcd}$$

Examples of natural transformations

- $$\begin{tikzcd} \mathbb{1} \arrow[r, bend left=50, "A"{name=U}] \arrow[r, bend right=50, "B"'{name=D}] & \mathcal{A} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$ $\alpha_* \quad A \longrightarrow B$ in $\mathcal{A}$ st $$\begin{tikzcd} A \arrow[r, "\alpha_*"] \arrow[d, "1_A"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "1_B"] \\ A \arrow[r, "\alpha_*"] & B \end{tikzcd}$$ ↙ vacuous condition

⇝ this is nothing but a morphism in $\mathcal{A}$ $\Rightarrow$ $\mathrm{Fun}(\mathbb{1}, \mathcal{A}) = \mathcal{A}$

- $$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left=50, "1_{\mathrm{Set}}"{name=U}] \arrow[r, bend right=50, "P"'{name=D}] & \mathrm{Set} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$ $$\alpha_X \quad \overset{X}{\overset{\shortparallel}{1_{\mathrm{Set}}(X)}} \longrightarrow P(X) \quad \text{in } \mathrm{Set}$$ $$x \longmapsto \{x\}$$

for all $f\ X \to Y$ in Set we have $$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & P(X) \arrow[d, "P(f)"] \\ Y \arrow[r, "\alpha_Y"] & P(Y) \end{tikzcd}$$,

Examples of natural transformations

- $$\begin{tikzcd} \mathbb{1} \arrow[r, bend left, "A"] \arrow[r, bend right, "B"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{A} \end{tikzcd}$$ $\alpha_* \; A \longrightarrow B$ in $\mathcal{A}$ st $$\begin{tikzcd} A \arrow[r, "\alpha_*"] \arrow[d, "1_A"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "1_B"] \\ A \arrow[r, "\alpha_*"] & B \end{tikzcd}$$ ↙ vacuous condition

⇝ this is nothing but a morphism in $\mathcal{A}$ $\Rightarrow$ $\mathrm{Fun}(\mathbb{1},\mathcal{A}) = \mathcal{A}$

- $$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "1_{\mathrm{Set}}"] \arrow[r, bend right, "P"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathrm{Set} \end{tikzcd}$$ $$\begin{array}{ccccc} \alpha_X & \overset{X}{\overset{\shortparallel}{1_{\mathrm{Set}}(X)}} & \longrightarrow & P(X) & \text{in Set} \\ & x & \longmapsto & \{x\} & \end{array}$$

for all $f\; X \rightarrow Y$ in Set we have $$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & P(X) \arrow[d, "P(f)"] \\ Y \arrow[r, "\alpha_Y"] & P(Y) \end{tikzcd}$$, indeed

$$\begin{tikzcd} x \in X \arrow[r, mapsto] \arrow[d, mapsto] & \{x\} \in P(X) \arrow[d, mapsto] \\ f(x) \in Y \arrow[r, mapsto] & \{f(x)\} \in P(Y) \end{tikzcd}$$

« Sameness » of functors

Let $\mathcal{A}, \mathcal{B}$ be categories and consider the functor category $\mathrm{Fun}(\mathcal{A},\mathcal{B})$

-

-

-

-

« Sameness » of functors

Let $\mathcal{A}, \mathcal{B}$ be categories and consider the functor category $\mathrm{Fun}(\mathcal{A},\mathcal{B})$

- A natural isomorphism is an isomorphism in $\mathrm{Fun}(\mathcal{A},\mathcal{B})$
-
-
-

« Sameness » of functors

Let $\mathcal{A}, \mathcal{B}$ be categories and consider the functor category $\mathrm{Fun}(\mathcal{A},\mathcal{B})$

- A natural isomorphism is an isomorphism in $\mathrm{Fun}(\mathcal{A},\mathcal{B})$
  $\alpha\ F \Rightarrow G$, $\alpha^{-1} G \Rightarrow F$ st $\alpha\alpha^{-1} = 1_G$, $\alpha^{-1}\alpha = 1_F$
-
-
-

« Sameness » of functors

Let $\mathcal{A}, \mathcal{B}$ be categories and consider the functor category $\mathrm{Fun}(\mathcal{A},\mathcal{B})$

- A natural isomorphism is an isomorphism in $\mathrm{Fun}(\mathcal{A},\mathcal{B})$

  $\alpha\ F \Rightarrow G\ ,\ \alpha^{-1}\ G \Rightarrow F$ st $\alpha\alpha^{-1} = 1_G\ ,\ \alpha^{-1}\alpha = 1_F$

- Two functors $F, G\ \mathcal{A} \longrightarrow \mathcal{B}$ are isomorphic if they are connected by a natural isomorphism $F \overset{\sim}{\Rightarrow} G$

-

-

« Sameness » of functors

Let $\mathcal{A}, \mathcal{B}$ be categories and consider the functor category $\mathrm{Fun}(\mathcal{A},\mathcal{B})$

- A natural isomorphism is an isomorphism in $\mathrm{Fun}(\mathcal{A},\mathcal{B})$

  $\alpha\ F \Rightarrow G\ ,\ \alpha^{-1}\ G \Rightarrow F$ st $\alpha\alpha^{-1} = 1_G\ ,\ \alpha^{-1}\alpha = 1_F$

- Two functors $F, G\ \mathcal{A} \longrightarrow \mathcal{B}$ are isomorphic if they are connected by a natural isomorphism $F \overset{\sim}{\Rightarrow} G$

- A natural transformation $\alpha\ F \Rightarrow G$ is a natural isomorphism if and only if $\alpha_A\ F(A) \longrightarrow G(A)$ is an isomorphism in $\mathcal{B}$ for all A

-

« Sameness » of functors

Let $\mathcal{A}, \mathcal{B}$ be categories and consider the functor category $\mathrm{Fun}(\mathcal{A},\mathcal{B})$

- A natural isomorphism is an isomorphism in $\mathrm{Fun}(\mathcal{A},\mathcal{B})$

  $\alpha\ F \Rightarrow G\ ,\ \alpha^{-1}\, G \Rightarrow F$ st $\alpha\alpha^{-1} = 1_G\ ,\ \alpha^{-1}\alpha = 1_F$

- Two functors $F, G\ \mathcal{A} \longrightarrow \mathcal{B}$ are isomorphic if they are connected by a natural isomorphism $F \overset{\sim}{\Rightarrow} G$

- A natural transformation $\alpha\ F \Rightarrow G$ is a natural isomorphism if and only if $\alpha_A\ F(A) \longrightarrow G(A)$ is an isomorphism in $\mathcal{B}$ for all A

- Example $1_{\mathrm{fdVS}_k} \overset{ev}{\Longrightarrow} (-)^{**}$ given, for all $V \in \mathrm{obj}(\mathrm{fdVS}_k)$, by

« Sameness » of functors

Let $\mathcal{A}, \mathcal{B}$ be categories and consider the functor category $\mathrm{Fun}(\mathcal{A},\mathcal{B})$

- A natural isomorphism is an isomorphism in $\mathrm{Fun}(\mathcal{A},\mathcal{B})$

  $\alpha\ F \Rightarrow G$, $\alpha^{-1}\ G \Rightarrow F$ st $\alpha\alpha^{-1} = 1_G$, $\alpha^{-1}\alpha = 1_F$

- Two functors $F, G\ \mathcal{A} \to \mathcal{B}$ are isomorphic if they are connected by a natural isomorphism $F \overset{\sim}{\Rightarrow} G$

- A natural transformation $\alpha\ F \Rightarrow G$ is a natural isomorphism if and only if $\alpha_A\ F(A) \longrightarrow G(A)$ is an isomorphism in $\mathcal{B}$ for all A

- Example $1_{\mathrm{fdVS}_k} \xRightarrow{ev} (-)^{**}$ given, for all $V \in \mathrm{obj}(\mathrm{fdVS}_k)$, by

  $$V \xrightarrow{ev^V} \mathrm{Hom}(V^*, k) = \mathrm{Hom}(\mathrm{Hom}(V,k), k)$$

  $$\longmapsto$$

« Sameness » of functors

Let $\mathcal{A}, \mathcal{B}$ be categories and consider the functor category $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$

- A natural isomorphism is an isomorphism in $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$

  $$\alpha \ F \Rightarrow G \ , \ \alpha^{-1} \ G \Rightarrow F \quad \text{st} \quad \alpha\alpha^{-1} = 1_G \ , \ \alpha^{-1}\alpha = 1_F$$

- Two functors $F, G \ \mathcal{A} \longrightarrow \mathcal{B}$ are isomorphic if they are connected by a natural isomorphism $F \overset{\sim}{\Rightarrow} G$

- A natural transformation $\alpha \ F \Rightarrow G$ is a natural isomorphism if and only if $\alpha_A \ F(A) \longrightarrow G(A)$ is an isomorphism in $\mathcal{B}$ for all A

- Example $\quad 1_{\mathrm{fdVS}_k} \xRightarrow{ev} (-)^{**}$ given, for all $V \in \mathrm{obj}(\mathrm{fdVS}_k)$, by

$$V \xrightarrow{\quad ev^V \quad} \mathrm{Hom}(V^*, k) = \mathrm{Hom}(\mathrm{Hom}(V, k), k)$$

$$v \longmapsto [\phi \ V \longrightarrow k] \xmapsto[ev^V_v]{} \phi(v)$$

«Sameness» of functors

Let $\mathcal{A}, \mathcal{B}$ be categories and consider the functor category $\mathrm{Fun}(\mathcal{A},\mathcal{B})$

- A natural isomorphism is an isomorphism in $\mathrm{Fun}(\mathcal{A},\mathcal{B})$

  $\alpha\ F \Rightarrow G\ ,\ \alpha^{-1}\ G \Rightarrow F$ st $\alpha\alpha^{-1} = 1_G\ ,\ \alpha^{-1}\alpha = 1_F$

- Two functors $F, G\ \mathcal{A} \to \mathcal{B}$ are isomorphic if they are connected by a natural isomorphism $F \overset{\cong}{\Rightarrow} G$

- A natural transformation $\alpha\ F \Rightarrow G$ is a natural isomorphism if and only if $\alpha_A\ F(A) \longrightarrow G(A)$ is an isomorphism in $\mathcal{B}$ for all A

- Example $\quad 1_{\mathrm{fdVS}_k} \xRightarrow{ev} (-)^{**}$ given, for all $V \in \mathrm{obj}(\mathrm{fdVS}_k)$, by

$$\begin{tikzcd} V \arrow[r, "ev^V", "\cong"'] & \mathrm{Hom}(V^*, k) = \mathrm{Hom}(\mathrm{Hom}(V,k), k) \\ v \arrow[r, mapsto] & {[\phi\ V \to k]} \arrow[r, mapsto, "ev^V_v"'] & \phi(v) \end{tikzcd}$$

Functors encoding the viewpoint of an object

> « A category is a world of objects, all looking at one another. Each sees the world from a different viewpoint »
>
> Tom Leinster

-

-

-

Functors encoding the viewpoint of an object

> « A category is a world of objects, all looking at one another Each sees the world from a different viewpoint »
>
> Tom Leinster

- The information of what an object A of a category $\mathcal{A}$ sees of $\mathcal{A}$ is codified by the functor $h^A \ \mathcal{A} \longrightarrow \mathrm{Set}$
-
-

Functors encoding the viewpoint of an object

> « A category is a world of objects, all looking at one another. Each sees the world from a different viewpoint »
>
> Tom Leinster

- The information of what an object A of a category $\mathcal{A}$ sees of $\mathcal{A}$ is codified by the functor $h^A \; \mathcal{A} \longrightarrow \mathrm{Set}$
- The information of what the category $\mathcal{A}$ sees of one of its objects A is codified by the functor $h_A \; \mathcal{A}^{op} \longrightarrow \mathrm{Set}$
-

Functors encoding the viewpoint of an object

> « A category is a world of objects, all looking at one another Each sees the world from a different viewpoint »
>
> Tom Leinster

- The information of what an object A of a category $\mathcal{A}$ sees of $\mathcal{A}$ is codified by the functor $h^A \; \mathcal{A} \longrightarrow \mathbf{Set}$
- The information of what the category $\mathcal{A}$ sees of one of its objects A is codified by the functor $h_A \; \mathcal{A}^{op} \longrightarrow \mathbf{Set}$
- We say that a functor $F \; \mathcal{A} \longrightarrow \mathbf{Set}$ (resp $F \; \mathcal{A}^{op} \longrightarrow \mathbf{Set}$) is representable if it is naturally isomorphic to a functor h^A (resp h_A) for some $A \in \mathrm{obj}(\mathcal{A})$

The Yoneda lemma

Let $\mathcal{A}$ be a locally small category. Then, we have that

$$\mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Set})(h_A, F) \cong F(A)$$

naturally in $A \in \mathrm{obj}(\mathcal{A})$ and $F \in \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Set})$

The Yoneda lemma

Let $\mathscr{A}$ be a locally small category Then, we have that

$$\mathrm{Fun}(\mathscr{A}^{op},\mathrm{Set})\,(h_A, F) \cong F(A)$$

naturally in $A \in \mathrm{ob}(\mathscr{A})$ and $F \in \mathrm{Fun}(\mathscr{A}^{op}, \mathrm{Set})$

The Yoneda lemma

Let $\mathscr{A}$ be a locally small category Then, we have that

$$\mathrm{Fun}(\mathscr{A}^{op}, \mathbf{Set})(h_A, F) \cong F(A)$$

naturally in $A \in \mathrm{ob}(\mathscr{A})$ and $F \in \mathrm{Fun}(\mathscr{A}^{op}, \mathbf{Set})$

The Yoneda lemma

Let $\mathcal{A}$ be a locally small category Then, we have that

$$\mathrm{Fun}(\mathcal{A}^{op},\mathbf{Set})(h_A, F) \cong F(A)$$

naturally in $A \in \mathrm{ob}(\mathcal{A})$ and $F \in \mathrm{Fun}(\mathcal{A}^{op},\mathbf{Set})$

Sketch of the proof

$$\mathrm{Fun}(\mathcal{A}^{op},\mathbf{Set})(h_A, F) \longrightarrow F(A)$$

$$F(A) \longrightarrow \mathrm{Fun}(\mathcal{A}^{op},\mathbf{Set})(h_A, F)$$

The Yoneda lemma

Let $\mathscr{A}$ be a locally small category Then, we have that

$$\mathrm{Fun}(\mathscr{A}^{op},\mathrm{Set})(h_A, F) \cong F(A)$$

naturally in $A \in \mathrm{obj}(\mathscr{A})$ and $F \in \mathrm{Fun}(\mathscr{A}^{op}, \mathrm{Set})$

Sketch of the proof

$$\mathrm{Fun}(\mathscr{A}^{op},\mathrm{Set})(h_A, F) \longrightarrow F(A)$$

$$\alpha \quad h_A \Rightarrow F \quad \longmapsto$$

$$\rightsquigarrow$$

$$h_A(A) = \mathscr{A}(A,A) \xrightarrow{\alpha_A} F(A)$$

$$F(A) \longrightarrow \mathrm{Fun}(\mathscr{A}^{op},\mathrm{Set})(h_A, F)$$

The Yoneda lemma

Let $\mathcal{A}$ be a locally small category Then, we have that

$$\mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})(h_A, F) \cong F(A)$$

naturally in $A \in \mathrm{obj}(\mathcal{A})$ and $F \in \mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})$

Sketch of the proof

$$\mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})(h_A, F) \longrightarrow F(A)$$

$$\alpha \quad h_A \Rightarrow F \longmapsto$$

$$\rightsquigarrow$$

$$\begin{tikzcd}
h_A(A) = \mathcal{A}(A,A) \arrow[r, "\alpha_A"] & F(A) \\
1_A \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \alpha_A(1_A) \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$$F(A) \longrightarrow \mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})(h_A, F)$$

The Yoneda lemma

Let $\mathcal{A}$ be a locally small category Then, we have that

$$\mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})(h_A, F) \cong F(A)$$

naturally in $A \in \mathrm{obj}(\mathcal{A})$ and $F \in \mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})$

Sketch of the proof

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})(h_A, F) \arrow[r] & F(A) \\
\alpha \quad h_A \Rightarrow F \arrow[r, mapsto] \arrow[d, squiggly] & \alpha_A(1_A) \\
h_A(A) = \mathcal{A}(A,A) \arrow[r, "\alpha_A"] & F(A) \arrow[u, squiggly] \\
1_A \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & \alpha_A(1_A) \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}$$

$$F(A) \longrightarrow \mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})(h_A, F)$$

The Yoneda lemma

Let $\mathcal{A}$ be a locally small category. Then, we have that

$$\mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})\,(h_A, F) \cong F(A)$$

naturally in $A \in \mathrm{ob}(\mathcal{A})$ and $F \in \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Set})$

Sketch of the proof

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})(h_A, F) \arrow[rr] & & F(A) \\
\alpha \quad h_A \Rightarrow F \arrow[rr, mapsto] \arrow[d, squiggly] & & \alpha_A(1_A) \\
h_A(A) = \mathcal{A}(A,A) \arrow[rr, "\alpha_A"] & & F(A) \arrow[u, squiggly] \\
\rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} \\
1_A \arrow[rr, mapsto] & & \alpha_A(1_A)
\end{tikzcd}$$

$$\begin{tikzcd}
F(A) \arrow[r] & \mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})(h_A, F) \\
x \arrow[r, mapsto] & \alpha \quad h_A \Rightarrow F
\end{tikzcd}$$

The Yoneda lemma

Let $\mathcal{A}$ be a locally small category Then, we have that

$$\mathrm{Fun}(\mathcal{A}^{op}, \mathbf{Set})(h_A, F) \cong F(A)$$

naturally in $A \in \mathrm{ob}(\mathcal{A})$ and $F \in \mathrm{Fun}(\mathcal{A}^{op}, \mathbf{Set})$

Sketch of the proof

$$\mathrm{Fun}(\mathcal{A}^{op}, \mathbf{Set})(h_A, F) \longrightarrow F(A)$$

$$\begin{tikzcd}
\alpha \; h_A \Rightarrow F \arrow[r, mapsto] \arrow[d, rightsquigarrow] & \alpha_A(1_A) \\
h_A(A) = \mathcal{A}(A,A) \arrow[r, "\alpha_A"] \arrow[d, phantom, sloped, "\ni"] & F(A) \arrow[u, rightsquigarrow] \arrow[d, phantom, sloped, "\ni"] \\
1_A \arrow[r, mapsto] & \alpha_A(1_A)
\end{tikzcd}$$

$$F(A) \longrightarrow \mathrm{Fun}(\mathcal{A}^{op}, \mathbf{Set})(h_A, F)$$

$$x \longmapsto \alpha \; h_A \Rightarrow F$$

$$\alpha_B \quad \mathcal{A}(B,A) \longrightarrow F(B)$$

$$\longmapsto$$

The Yoneda lemma

Let $\mathcal{A}$ be a locally small category. Then, we have that

$$\mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})(h_A, F) \cong F(A)$$

naturally in $A \in \mathrm{obj}(\mathcal{A})$ and $F \in \mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})$

Sketch of the proof

$$\mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})(h_A, F) \longrightarrow F(A)$$

$$\alpha \quad h_A \Rightarrow F \longmapsto \alpha_A(1_A)$$

$$\rightsquigarrow \qquad\qquad\qquad \rightsquigarrow$$

$$\begin{tikzcd} h_A(A)=\mathcal{A}(A,A) \arrow[r, "\alpha_A"] & F(A) \\ 1_A \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \alpha_A(1_A) \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

$$F(A) \longrightarrow \mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})(h_A, F)$$

$$x \longmapsto \alpha \quad h_A \Rightarrow F$$

$$\alpha_B \quad \mathcal{A}(B,A) \longrightarrow F(B)$$

$$f \; B \rightarrow A \longmapsto$$

The Yoneda lemma

Let $\mathcal{A}$ be a locally small category Then, we have that

$$\mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})(h_A, F) \cong F(A)$$

naturally in $A \in \mathrm{ob}(\mathcal{A})$ and $F \in \mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})$

Sketch of the proof

$$\mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})(h_A, F) \longrightarrow F(A)$$

$$\begin{tikzcd}
\alpha \quad h_A \Rightarrow F \arrow[rr, mapsto] \arrow[d, squiggly] & & \alpha_A(1_A) \\
h_A(A) = \mathcal{A}(A,A) \arrow[rr, "\alpha_A"] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & F(A) \arrow[u, squiggly] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
1_A \arrow[rr, mapsto] & & \alpha_A(1_A)
\end{tikzcd}$$

$$F(A) \longrightarrow \mathrm{Fun}(\mathcal{A}^{op},\mathrm{Set})(h_A, F)$$

$$x \longmapsto \alpha \quad h_A \Rightarrow F$$

$$\alpha_B \quad \mathcal{A}(B,A) \longrightarrow F(B)$$

$$\begin{tikzcd}
f \quad B \to A \arrow[r, mapsto] \arrow[d, squiggly] & {} \\
F(A) \arrow[r, "F(f)"] & F(B)
\end{tikzcd}$$

The Yoneda lemma

Let $\mathcal{A}$ be a locally small category Then, we have that

$$\mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Set})(h_A, F) \cong F(A)$$

naturally in $A \in \mathrm{obj}(\mathcal{A})$ and $F \in \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Set})$

Sketch of the proof

$$\mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Set})(h_A, F) \longrightarrow F(A)$$

$$\alpha \; h_A \Rightarrow F \longmapsto \alpha_A(1_A)$$

$$\rightsquigarrow \qquad\qquad \nearrow$$

$$\begin{tikzcd} h_A(A) = \mathcal{A}(A,A) \arrow[r, "\alpha_A"] & F(A) \\ 1_A \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \alpha_A(1_A) \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

$$F(A) \longrightarrow \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Set})(h_A, F)$$

$$x \longmapsto \alpha \; h_A \Rightarrow F$$

$$\alpha_B \quad \mathcal{A}(B,A) \longrightarrow F(B)$$

$$f \; B \to A \longmapsto F(f)(x)$$

$$\searrow \qquad\qquad \nearrow$$

$$F(A) \xrightarrow{F(f)} F(B)$$

The Yoneda embedding

Let $\mathcal{A}$ be a locally small category. The functor

$$\begin{tikzcd} \mathcal{A} \arrow[r, "Y"] & \mathrm{Fun}(\mathcal{A}^{op}, \mathbf{Set}) \end{tikzcd}$$

$$\longmapsto$$

$$\overset{\circledast}{\longmapsto}$$

is fully faithful

The Yoneda embedding

Let $\mathcal{A}$ be a locally small category. The functor

$$\begin{tikzcd} \mathcal{A} \arrow[r, "y"] & \mathrm{Fun}(\mathcal{A}^{op}, \mathcal{S}et) \\ A \arrow[r, mapsto] & h_A \end{tikzcd}$$

$$\overset{\circledast}{\longmapsto}$$

is fully faithful

The Yoneda embedding

Let $\mathcal{A}$ be a locally small category The functor

$$\mathcal{A} \xrightarrow{Y} \mathrm{Fun}(\mathcal{A}^{op}, \mathbf{Set})$$

$$A \longmapsto h_A$$

$$A \xrightarrow{f} B \overset{\circledast}{\longmapsto} h_f \quad h_A \Rightarrow h_B$$ given, for all $C \in obj(\mathcal{A})$, by

$$(h_f)_C \quad \mathcal{A}(C,A) \xrightarrow{f \circ -} \mathcal{A}(C,B)$$

is fully faithful

The Yoneda embedding

Let $\mathcal{A}$ be a locally small category The functor

$$\mathcal{A} \xrightarrow{\mathcal{Y}} \mathrm{Fun}(\mathcal{A}^{op}, \mathbf{Set})$$

$$A \longmapsto h_A$$

$$A \xrightarrow{f} B \overset{\circledast}{\longmapsto} h_f \; h_A \Rightarrow h_B$$ given, for all $C \in obj(\mathcal{A})$, by

$$(h_f)_C \quad \mathcal{A}(C,A) \xrightarrow{f\circ -} \mathcal{A}(C,B) \quad [g \; C \to A] \longmapsto [fg \; C \to B]$$

is fully faithful

The Yoneda embedding

Let $\mathcal{A}$ be a locally small category The functor

$$\mathcal{A} \xrightarrow{Y} \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Set})$$

$$A \longmapsto h_A$$

$$A \xrightarrow{f} B \overset{\circledast}{\longmapsto} h_f \; h_A \Rightarrow h_B$$ given, for all $C \in \mathrm{obj}(\mathcal{A})$, by

$$(h_f)_C \;\; \mathcal{A}(C,A) \xrightarrow{f \circ -} \mathcal{A}(C,B) \quad [g \; C \to A] \longmapsto [fg \; C \to B]$$

is fully faithful

Sketch of the proof

Yoneda lemma gives us a bijection $F(A) \longrightarrow \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Set})(h_A, F)$

The Yoneda embedding

Let $\mathcal{A}$ be a locally small category The functor

$$\mathcal{A} \xrightarrow{y} \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Set})$$

$$A \longmapsto h_A$$

$$A \xrightarrow{f} B \overset{\circledast}{\longmapsto} h_f \;\; h_A \Rightarrow h_B$$ given, for all $C \in \mathrm{obj}(\mathcal{A})$, by

$$(h_f)_C \;\; \mathcal{A}(C,A) \xrightarrow{f\circ -} \mathcal{A}(C,B) \;\; [g \;\; C \to A] \longmapsto [fg \;\; C \to B]$$

is fully faithful

Sketch of the proof

Yoneda lemma gives us a bijection $F(A) \longrightarrow \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Set})(h_A, F)$

Take $F = h_B \Rightarrow$

The Yoneda embedding

Let $\mathcal{A}$ be a locally small category The functor

$$\begin{array}{ccl} \mathcal{A} & \xrightarrow{\ y\ } & \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Set}) \\ A & \longmapsto & h_A \\ A \xrightarrow{f} B & \overset{\circledast}{\longmapsto} & h_f \; h_A \Rightarrow h_B \end{array}$$

given, for all $C \in \mathrm{obj}(\mathcal{A})$, by

$$(h_f)_C \quad \mathcal{A}(C,A) \xrightarrow{f\circ -} \mathcal{A}(C,B) \quad [g \; C \to A] \longmapsto [fg \; C \to B]$$

is fully faithful

Sketch of the proof

Yoneda lemma gives us a bijection $F(A) \longrightarrow \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Set})(h_A, F)$

Take $F = h_B \Rightarrow \quad h_B(A) = \mathcal{A}(A,B) \longrightarrow \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Set})(h_A, h_B)$ which is precisely the function $\circledast$

Studying the inaccessible

An object A in a category $\mathcal{A}$ is fully determined by what the category sees of it

that is,

Studying the inaccessible

An object A in a category $\mathcal{A}$ is fully determined by what the category sees of it

that is,

it is fully determined by h_A

Studying the inaccessible

An object A in a category $\mathcal{A}$ is fully determined by what the category sees of it

that is,

it is fully determined by h_A

Indeed, if $\mathcal{A}$ sees two objects A and B as indistinguishable

that is,

Studying the inaccessible

An object A in a category $\mathcal{A}$ is fully determined by what the category sees of it

that is,

it is fully determined by h_A

Indeed, if $\mathcal{A}$ sees two objects A and B as indistinguishable

that is,

$h_A \cong h_B$

Studying the inaccessible

An object A in a category $\mathcal{A}$ is fully determined by what the category sees of it

that is,

it is fully determined by h_A

Indeed, if $\mathcal{A}$ sees two objects A and B as indistinguishable

that is,

$h_A \cong h_B$

then, A and B are the «same» in $\mathcal{A}$

that is,

Studying the inaccessible

An object A in a category $\mathcal{A}$ is fully determined by what the category sees of it

that is,

it is fully determined by h_A

Indeed, if $\mathcal{A}$ sees two objects A and B as indistinguishable

that is,

$h_A \cong h_B$

then, A and B are the «same» in $\mathcal{A}$

that is,

$A \cong B$

Studying the inaccessible

In general, in order to fully characterize an object $A \in \mathrm{obj}(\mathcal{A})$ need all the information of h_A

Studying the inaccessible

In general, in order to fully characterize an object $A \in obj(\mathcal{A})$ need all the information of h_A

But there are certain categories were less is required

Studying the inaccessible

In general, in order to fully characterize an object $A \in \mathrm{obj}(\mathcal{A})$ need all the information of h_A

But there are certain categories were less is required

$X \in \mathrm{obj}(\mathbf{Set})$ is fully determined by $\mathbf{Set}(\{*\}, X)$

$$\{*\} \xrightarrow{f} X \quad \equiv \quad x \in X$$

Studying the inaccessible

In general, in order to fully characterize an object $A \in obj(\mathcal{A})$ need all the information of h_A

But there are certain categories were less is required

$X \in obj(\mathbf{Set})$ is fully determined by $\mathbf{Set}(\{*\}, X)$

$$\{*\} \xrightarrow{f} X \quad \equiv \quad x \in X$$

Sets are determined by their points!

Moduli spaces

- A moduli space classifying certain objects (the real numbers, triangles, vector bundles on a manifold) is a space (topological space, manifold, abelian variety, scheme, stack) in which each point represents one object, two non-isomorphic objects are represented by different points and objects that are «similar» are «closeby» in this space

Moduli spaces

- A moduli space classifying certain objects (the real numbers, triangles, vector bundles on a manifold) is a space (topological space, manifold, abelian variety, scheme, stack) in which each point represents one object, two non-isomorphic objects are represented by different points and objects that are «similar» are «closeby» in this space

Example the real line $\mathbb{R}$ is the moduli space classifying the real numbers

Moduli spaces

- Problem We want to study a moduli space M classifying certain objects what does it look like?

Moduli spaces

- **Problem** We want to study a moduli space M classifying certain objects. What does it look like?

$\{*\} \longrightarrow M$ sees the points of M

Moduli spaces

- **Problem** We want to study a moduli space M classifying certain objects. What does it look like?

$\{*\} \longrightarrow M$ sees the points of M

$[0,1] \longrightarrow M$ sees the paths of M

Moduli spaces

- **Problem** We want to study a moduli space M classifying certain objects. What does it look like?

$$\left.\begin{array}{lll} \{*\} \longrightarrow M & & \text{sees the points of } M \\[1em] [0,1] \longrightarrow M & & \text{sees the paths of } M \\[1em] \mathbb{R} \longrightarrow M & & \text{sees the curves of } M \end{array}\right\}$$

Moduli spaces

- Problem We want to study a moduli space M classifying certain objects. What does it look like?

$$\left.\begin{array}{ll} \{*\} \longrightarrow M & \text{sees the points of } M \\ [0,1] \longrightarrow M & \text{sees the paths of } M \\ \mathbb{R} \longrightarrow M & \text{sees the curves of } M \end{array}\right\} \text{in a certain category of spaces } \mathbf{Spac}$$

Moduli spaces

- **Problem** We want to study a moduli space M classifying certain objects what does it look like?

$\{*\} \longrightarrow M$ sees the points of M

$[0,1] \longrightarrow M$ sees the paths of M

$\mathbb{R} \longrightarrow M$ sees the curves of M

in a certain category of spaces **Spac**

What Yoneda lemma is telling us is that M is fully determined by what the other spaces see of M

More general spaces

We want to build on M from what all the spaces in our category Spac see

$$\begin{aligned} \mathcal{M} \quad \mathrm{Spac}^{op} &\longrightarrow \mathrm{Set} \\ X &\longmapsto \text{what } X \text{ sees of } M \end{aligned}$$

More general spaces

We want to build an M from what all the spaces in our category Spac see

$$\begin{array}{rcl} \mathcal{M} \quad \mathsf{Spac}^{op} & \longrightarrow & \mathsf{Set} \\ X & \longmapsto & \text{what } X \text{ sees of } M \end{array}$$

if $\mathcal{M}$ is representable $\mathcal{M} \overset{\sim}{\Rightarrow} h_M$ we obtain our space

More general spaces

We want to build an M from what all the spaces in our category Spac see

$$\begin{aligned} \mathcal{M} \quad \mathrm{Spac}^{op} &\longrightarrow \mathrm{Set} \\ X &\longmapsto \text{what } X \text{ sees of } M \end{aligned}$$

if $\mathcal{M}$ is representable $\mathcal{M} \overset{\sim}{\Rightarrow} h_M$ we obtain our space

what if not?

More general spaces

We want to build an M from what all the spaces in our category Spac see

$$\begin{aligned} \mathcal{M} \quad \mathrm{Spac}^{op} &\longrightarrow \mathrm{Set} \\ X &\longmapsto \text{what } X \text{ sees of } M \end{aligned}$$

If $\mathcal{M}$ is representable $\mathcal{M} \overset{\sim}{\Rightarrow} h_M$ we obtain our space

What if not? Just consider $\mathcal{M}$ as a space itself!

More general spaces

We want to build an M from what all the spaces in our category Spac see

$$\mathcal{M} \quad \mathrm{Spac}^{op} \longrightarrow \mathrm{Set}$$
$$X \longmapsto \text{what } X \text{ sees of } M$$

If $\mathcal{M}$ is representable $\mathcal{M} \overset{\sim}{\Rightarrow} h_M$ we obtain our space

What if not? Just consider $\mathcal{M}$ as a space itself!

We do not see it, but we know how it is perceived by different spaces, and this is often enough to do geometry!

Thank you for your attention

References

BOOKS

[Leinster, T] Basic Category Theory

[Maclane, S] Categories for the Working Mathematician

[Riehl, E] Category Theory in Context

BLOG POST

[Ward, M] The Brilliance of the Yoneda Lemma